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LETTER TO THE EDITOR

Models for equilibrium BEC superradiance

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Abstract

Motivated by recent experiments with superradiant Bose–Einstein condensate (BEC) we consider simple microscopic models describing rigorously the interference of the two cooperative phenomena, BEC and radiation, in thermodynamic equilibrium. Our results in equilibrium confirm the presence of the observed superradiant light scattering from BEC: (a) the equilibrium superradiance exists only below a certain transition temperature; (b) there is superradiance and matter–wave (BEC) enhancement due to the coherent coupling between light and matter.

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1. This letter is motivated by the interest in Bose–Einstein condensation (BEC) of bosons in *traps* and in particular, by the recent discovery of the Dicke superradiance and BEC matter–wave *amplification*, see e.g. [1–3]. In all these experiments, the condensate was illuminated with a Q-mode laser beam (the so-called 'dressing beam') and then the BEC atoms scatter photons from this beam into another mode and receive the corresponding recoil momentum producing a coherent 'four-wave mixing' of light and atoms [3]. Note that Girardeau in 1978, [11], had already anticipated the possible existence of these phenomena.

The irradiation of a Bose–Einstein condensate can be considered as an external action on it. The effects of other external agents have been studied, for instance the influence of boundary conditions [4, 5] and that of an external field, scaled with the volume so as to retain the space homogeneity [6]. In the first case it is well known that attractive boundary conditions enhance condensation. Here condensation occurs even in one dimension and is a consequence of a discrete point in the spectrum. In the other case one also finds an enhancement of condensation compared to that of the free Bose gas depending on the behaviour of the potential.

In this letter we consider *two* simple models which show that some particular interactions with a one-mode radiation can *enhance* the conventional BEC of the perfect Bose gas (PBG). This can be interpreted as a coherent coexistence of the BEC and the *condensation* of photons, which is a type of *equilibrium* Dicke superradiance induced by BEC, see e.g. [7–9].

We consider a system of non-interacting bosons (PBG) of mass m enclosed in a cubic box $\Lambda \subset \mathbb{R}^d$ of volume $V = |\Lambda| = L^d$ centred at the origin with periodic boundary conditions. The Hamiltonian for the PBG is

$$T_\Lambda := \sum_{k\in\Lambda^*} \epsilon_k a_k^* a_k \tag{1}$$

where

$$k \in \Lambda^* = \left\{ k \in \mathbb{R}^d : k = \frac{2\pi n}{L}, n \in \mathbb{Z}^d \right\}$$

$\epsilon_k = \hbar^2 k^2/2m$ is the kinetic energy of one particle and a_k^* and a_k are the usual boson creation and annihilation operators corresponding to momentum $\hbar k$, satisfying the commutation relations

$$[a_k, a_{k'}^*] = \delta_{kk'} \qquad [a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0.$$

In both of our models the PBG (1) interacts with a one-mode *photon* field with Hamiltonian

$$\Omega b^* b \tag{2}$$

where b^* and b are the photon creation and annihilation operators, satisfying $[b, b^*] = 1$, and $\Omega > 0$ is the energy of a single photon.

In both our models we assume that the photons interact with bosons *linearly*. The interaction in the first model has the form

$$U_{1\Lambda} = \tfrac{1}{2} g_1 (a_Q^* b + a_Q b^*) \tag{3}$$

whereas in the second one it is

$$U_{2\Lambda} = \tfrac{1}{2} g_2 (a_{Q'}^* b^* + a_{Q'} b). \tag{4}$$

Hence the Hamiltonians of our models have the form

$$H_{1,2\Lambda} = \tilde{T}_\Lambda + \Omega b^* b + U_{1,2\Lambda} \tag{5}$$

where $\tilde{T}_\Lambda := \sum_{k\in\Lambda, k\neq 0} \epsilon_k a_k^* a_k$ and $g_{1,2} \geqslant 0$.

The different forms of interactions (3) and (4) come from two possible mechanisms of the light-bosons couplings.

It is known that in the cubic box the *conventional* BEC of the PBG occurs only in the mode $k = 0$, see e.g. [10]. Since the scattering of the Q-mode light is most important on the macroscopic amount of condensed particles, following [11, 12] we retain in the interaction only terms representing excitation from, and de-excitation back, to the BEC (in conformity with the minimal coupling in electrodynamics)

$$\widehat{U}_{1\Lambda} = \frac{1}{2} \frac{\lambda_1}{\sqrt{V}} (a_Q^* a_0 b_Q + a_0^* a_Q b_Q^*) \tag{6}$$

with a coupling $\lambda_1 \geqslant 0$. Now assuming (as in [3]) that the depletion of the condensate can be neglected, one can simplify (6) by substituting

$$\frac{1}{\sqrt{V}} a_0 \to \sqrt{\rho_0}\, \mathrm{e}^{\mathrm{i}\varphi} \qquad \frac{1}{\sqrt{V}} a_0^* \to \sqrt{\rho_0}\, \mathrm{e}^{-\mathrm{i}\varphi} \tag{7}$$

where ρ_0 is the $k = 0$ mode condensate density. This leads to the interaction (3), with the coupling constant $g_1 = \lambda_1\sqrt{\rho_0}$, after a gauge transformation eliminating the zero mode condensate phase φ and after putting $b_Q = b$.

The interaction (4) has as its origin in the 'four-wave mixing' mechanism , see e.g. [1–3]. A condensate illuminated by a Q-mode 'dressing' laser beam ('dressed condensate') can spontaneously emit pairs of photons and recoiling atoms. The simplest way to take this into account is described by the Hamiltonian [3]

$$\widehat{U}_{2\Lambda} = \frac{1}{2}\frac{\lambda_2}{\sqrt{V}}(a^*_{Q'}b^*_{Q''}a_0 b_Q + a^*_0 b^*_Q a_{Q'} b_{Q''}) \tag{8}$$

where $b^*_{Q''}$, $b_{Q''}$ correspond to superradiated photons with a wave-vector $Q'' = Q - Q'$. If as in [3] one neglects the depletion of the $k = 0$ mode 'dressed' condensate and the Q-mode 'dressing' laser beam, then as in (7) the substitution of the corresponding operators by c-numbers gives instead of (8) the interaction (4). Now g_2 is proportional to $\lambda_2\sqrt{\rho_0}$ and the amplitude of the 'dressing' laser beam, and $b_{Q''} = b$.

The aim of this letter is to study the thermodynamic *equilibrium* properties of the models (5) and the possible *phase transitions* due to the coherent interaction of *recoiled* condensate atoms with scattered (*superradiated*) photons.

2. These models can be solved *exactly* by canonical transformations diagonalizing the Hamiltonians (5). To establish the thermodynamic properties we consider the *grand-canonical ensemble* Hamiltonians

$$H_{1,2\Lambda}(\mu) = H_{1,2\Lambda} - \mu\tilde{N}_\Lambda \tag{9}$$

where μ is the chemical potential and

$$\tilde{N}_\Lambda = \sum_{k\in\Lambda, k\neq 0} a^*_k a_k \tag{10}$$

is the particle number operator for the system excluding the ground state condensate. Since we shall follow a procedure analogous to that used for the PBG it is useful to recall some facts about the latter.

For the PBG the finite-volume grand-canonical thermodynamic functions exist only for $\mu < 0$. Denoting the *grand-canonical Gibbs state* corresponding to the Hamiltonian T_Λ at inverse temperature β by $\langle - \rangle_{T_\Lambda}(\mu)$, we have for $\mu < 0$,

$$\lim_{V\to\infty}\left\langle \frac{N_\Lambda}{V}\right\rangle_{T_\Lambda}(\mu) = \rho_0(\mu) \tag{11}$$

where $N_\Lambda = \tilde{N}_\Lambda + a^*_0 a_0$ and

$$\rho_0(\mu) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\mathrm{d}^d k}{\mathrm{e}^{\beta(\epsilon_k-\mu)} - 1}. \tag{12}$$

For $d \geqslant 3$, $\rho_{0c} \equiv \rho_0(0) < \infty$ and therefore, to be able to access densities higher than ρ_{0c} in this case, one has to work with a given density rather than the chemical potential. We fix the finite-volume chemical potential μ_V by the equation

$$\rho = \left\langle \frac{N_\Lambda}{V}\right\rangle_{T_\Lambda}(\mu_V). \tag{13}$$

One then finds that for $\rho \leqslant \rho_{0c}$, $\mu_V \to \mu_\infty$ where $\mu_\infty < 0$ is the unique solution of the equation

$$\rho = \rho_0(\mu). \tag{14}$$

On the other hand for $\rho > \rho_{0c}$,

$$\mu_V = -\frac{1}{\beta V(\rho - \rho_{0c})} + \mathrm{O}(1/V^2). \tag{15}$$

For the expectation of the zero mode occupation particle density we have

$$\lim_{V\to\infty} \left\langle \frac{a_0^* a_0}{V} \right\rangle_{T_\Lambda} (\mu_V) = \begin{cases} 0 & \text{for} \quad \rho \leqslant \rho_{0c} \\ \rho - \rho_{0c} & \text{for} \quad \rho > \rho_{0c}. \end{cases} \tag{16}$$

2.1. Returning to our first model $H_{1\Lambda}$ let us now define new boson operators ξ_1^*, ξ_1 and η_1^*, η_1 by the canonical transformation

$$\begin{aligned} \xi_1 &= \cos\vartheta\, a_Q + \sin\vartheta\, b \\ \eta_1 &= \sin\vartheta\, a_Q - \cos\vartheta\, b \end{aligned} \tag{17}$$

where ϑ satisfies the equation

$$\tan 2\vartheta = -\frac{g_1}{\Omega - \epsilon_Q + \mu}. \tag{18}$$

In terms of these operators we can write the Hamiltonian $H_{1\Lambda}(\mu)$ in the form

$$H_{1\Lambda}(\mu) = \sum_{k\in\Lambda^*, k\neq 0, Q} (\epsilon_k - \mu) a_k^* a_k + E_{1+}(\mu, \Omega)\xi_1^*\xi_1 + E_{1-}(\mu, \Omega)\eta_1^*\eta_1 \tag{19}$$

where

$$\begin{aligned} E_{1+}(\mu, \Omega) &= \tfrac{1}{2}\left\{\Omega + \epsilon_Q - \mu + \sqrt{(\Omega + \mu - \epsilon_Q)^2 + g_1^2}\right\} \\ E_{1-}(\mu, \Omega) &= \tfrac{1}{2}\left\{\Omega + \epsilon_Q - \mu - \sqrt{(\Omega + \mu - \epsilon_Q)^2 + g_1^2}\right\}. \end{aligned} \tag{20}$$

Since $E_{1+}(\mu, \Omega) \geqslant E_{1-}(\mu, \Omega)$, the thermodynamic stability of the Hamiltonian $H_{1\Lambda}(\mu)$ requires that

$$E_{1-}(\mu, \Omega) \geqslant 0 \qquad \text{or} \qquad \mu \leqslant \mu_c(g_1, \Omega) \tag{21}$$

where

$$\mu_c(g, \Omega) \equiv \min\left\{0, -\frac{g_1^2}{4\Omega} + \epsilon_Q\right\}. \tag{22}$$

Denoting the *grand-canonical Gibbs state* corresponding to the Hamiltonian $H_{1\Lambda}$ by $\langle - \rangle_{H_{1\Lambda}}(\mu)$, we have for $\mu < \mu_c$,

$$\begin{aligned} \left\langle \frac{a_Q^* a_Q}{V} \right\rangle_{H_{1\Lambda}} (\mu) = \frac{1}{2V} \Bigg\{ &\left(\frac{1}{\mathrm{e}^{\beta E_{1+}} - 1} + \frac{1}{\mathrm{e}^{\beta E_{1-}} - 1} \right) \\ &- \frac{\Omega + \mu - \epsilon_Q}{E_{1+} - E_{1-}} \left(\frac{1}{\mathrm{e}^{\beta E_{1+}} - 1} - \frac{1}{\mathrm{e}^{\beta E_{1-}} - 1} \right) \Bigg\} \end{aligned} \tag{23}$$

and

$$\left\langle \frac{a_k^* a_k}{V} \right\rangle_{H_{1\Lambda}} (\mu) = \frac{1}{V} \frac{1}{\mathrm{e}^{\beta(\epsilon_k - \mu)} - 1} \tag{24}$$

for $k \neq 0, Q$. Thus, since $E_{1-}(\mu, \Omega) > 0$ for $\mu < \mu_c$, we obtain for the total particle density

$$\lim_{V\to\infty} \left\langle \frac{\tilde{N}_\Lambda}{V} \right\rangle_{H_{1\Lambda}} (\mu) = \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda, k\neq 0} \frac{1}{\mathrm{e}^{\beta(\epsilon_k - \mu)} - 1} = \rho_0(\mu). \tag{25}$$

Therefore, as in the PBG we have to fix the finite-volume chemical potential μ_V by the equation

$$\rho = \left\langle \frac{\tilde{N}_\Lambda}{V} \right\rangle_{H_{1\Lambda}} (\mu_V) \tag{26}$$

and to distinguish two cases, $\rho \leqslant \rho_c$ and $\rho > \rho_c$ where

$$\rho_c \equiv \rho_0(\mu_c). \tag{27}$$

Note that by (12) if $\mu_c < 0$, then $\rho_c < \infty$ even for $d = 1, 2$.

(a) *Case*: $\rho \leqslant \rho_c$. Using (26) we see that in this case $\mu_V \to \mu_\infty$, where $\mu_\infty < \mu_c$ is the unique solution of

$$\rho = \rho_0(\mu). \tag{28}$$

Since for these values of μ, $E_{1-}(\mu, \Omega)$ is bounded below away from zero, by (23) we also obtain for the expectation of the Q mode occupation particle density

$$\lim_{V\to\infty} \left\langle \frac{a_Q^* a_Q}{V} \right\rangle_{H_{1\Lambda}} (\mu_V) = 0. \tag{29}$$

Similarly we obtain

$$\lim_{V\to\infty} \left\langle \frac{b^* b}{V} \right\rangle_{H_{1\Lambda}} (\mu_V) = \lim_{V\to\infty} \frac{1}{2V} \left\{ \left(\frac{1}{\mathrm{e}^{\beta E_{1+}} - 1} + \frac{1}{\mathrm{e}^{\beta E_{1-}} - 1} \right) \right.$$
$$\left. + \frac{\Omega + \mu - \epsilon_Q}{E_{1+} - E_{1-}} \left(\frac{1}{\mathrm{e}^{\beta E_{1+}} - 1} - \frac{1}{\mathrm{e}^{\beta E_{1-}} - 1} \right) \right\} = 0 \tag{30}$$

for the density of photons.

(b) *Case*: $\rho > \rho_c$. As above the analysis is similar to that for the PBG. Using (23)–(26) we see that for $V \to \infty$, μ_V takes the form

$$\mu_\Lambda = \mu_c - \frac{1}{V} \frac{1}{\beta(\rho - \rho_c)} + \mathrm{O}(1/V^2) \tag{31}$$

and

$$E_{1-}(\mu_V, \Omega) = \frac{1}{V} \frac{1}{\beta(\rho - \rho_c)} \frac{\Omega}{\Omega - \mu_c + \epsilon_Q} + \mathrm{O}(1/V^2). \tag{32}$$

Then by (23) and (32) we obtain instead of zero (see (29))

$$\lim_{V\to\infty} \left\langle \frac{a_Q^* a_Q}{V} \right\rangle_{H_{1\Lambda}} (\mu_V) = \rho - \rho_c \tag{33}$$

implying the occurrence of BEC of the Bose gas in the Q-mode. Similarly, instead of zero as in (30) we obtain

$$\lim_{V\to\infty} \left\langle \frac{b^* b}{V} \right\rangle_{H_{1\Lambda}} (\mu_V) = \frac{g_1^2}{4\Omega^2} (\rho - \rho_c) \tag{34}$$

that is, one has *condensation of photons* occurring simultaneously with BEC (33). This correlation can also be seen from the limit of the boson–photon correlation (*entangling*) function

$$\lim_{V\to\infty} \left\langle \frac{a_Q^* b}{V} \right\rangle_{H_{1\Lambda}} (\mu_V) = -\lim_{V\to\infty} \frac{g_1}{2V(\Omega + \mu_V - \epsilon_Q)} \{ (\mathrm{e}^{\beta E_{1+}} - 1)^{-1} - (\mathrm{e}^{\beta E_{1-}} - 1)^{-1} \}$$
$$= \begin{cases} -g_1(\rho - \rho_c)/2\Omega & \text{for} \quad \rho > \rho_c \\ 0 & \text{for} \quad \rho \leqslant \rho_c. \end{cases} \tag{35}$$

2.2. Now we consider our second model $H_{2\Lambda}$. We define new boson operators ξ_2^*, ξ_2 and η_2^*, η_2 by the canonical Bogoliubov transformation

$$\begin{aligned} \xi_2 &= \cosh\phi\, a_{Q'} - \sinh\phi\, b \\ \eta_2 &= \cosh\phi\, b - \sinh\phi\, a_{Q'} \end{aligned} \tag{36}$$

where ϕ satisfies the equation

$$\tanh 2\phi = \frac{g_2}{\Omega + \epsilon_{Q'} - \mu}. \tag{37}$$

In terms of these operators we can write the Hamiltonian $H_{2\Lambda}(\mu)$ in the form

$$\begin{aligned} H_{2\Lambda}(\mu) = \sum_{k\in\Lambda^*, k\neq 0, Q'} &(\epsilon_k - \mu)a_k^* a_k + E_{2+}(\mu,\Omega)\xi_2^*\xi_2 + E_{2-}(\mu,\Omega)\eta_2^*\eta_2 \\ &+ \frac{1}{2}\{(E_{2-} + E_{2-}) - (\epsilon_{Q'} - \mu + \Omega)\} \end{aligned} \tag{38}$$

where

$$\begin{aligned} E_{2+}(\mu,\Omega) &= \tfrac{1}{2}\Big\{(\epsilon_{Q'} - \mu - \Omega) + \sqrt{(\epsilon_{Q'} - \mu + \Omega)^2 - g_2^2}\Big\} \\ E_{2-}(\mu,\Omega) &= \tfrac{1}{2}\Big\{ - (\epsilon_{Q'} - \mu - \Omega) + \sqrt{(\epsilon_{Q'} - \mu + \Omega)^2 - g_2^2}\Big\}. \end{aligned} \tag{39}$$

The thermodynamic stability of the Hamiltonian $H_{2\Lambda}(\mu)$, always gives

$$\mu \leqslant \mu_c(g_2, \Omega) \tag{40}$$

where

$$\mu_c(g_2,\Omega) \equiv \min\left\{0, -\frac{g_2^2}{4\Omega} + \epsilon_{Q'}\right\}. \tag{41}$$

As in (27) we let

$$\rho_c \equiv \rho(\mu_c). \tag{42}$$

We have for $\mu < \mu_c$,

$$\begin{aligned} \langle a_{Q'}^* a_{Q'}\rangle_{H_{2\Lambda}}(\mu) = \frac{1}{2}\Bigg\{&\left(\frac{1}{e^{\beta E_{2+}} - 1} - \frac{1}{e^{\beta E_{2-}} - 1}\right) + \frac{\epsilon_{Q'} - \mu + \Omega}{E_{2+} + E_{2-}} \\ &\times\left(\frac{1}{e^{\beta E_{2+}} - 1} + \frac{1}{e^{\beta E_{2-}} - 1}\right) + \frac{\epsilon_{Q'} - \mu + \Omega}{E_{2+} + E_{2-}} - 1\Bigg\} \end{aligned} \tag{43}$$

$$\langle a_k^* a_k\rangle_{H_{2\Lambda}}(\mu) = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \qquad \text{for} \quad k \neq 0, Q' \tag{44}$$

$$\begin{aligned} \langle b^* b\rangle_{H_{2\Lambda}}(\mu) = \frac{1}{2}\Bigg\{&-\left(\frac{1}{e^{\beta E_{2+}} - 1} - \frac{1}{e^{\beta E_{2-}} - 1}\right) + \frac{\epsilon_{Q'} - \mu + \Omega}{E_{2+} + E_{1-}} \\ &\times\left(\frac{1}{e^{\beta E_{2+}} - 1} + \frac{1}{e^{\beta E_{2-}} - 1}\right) + \frac{\epsilon_{Q'} - \mu + \Omega}{E_{2+} + E_{1-}} - 1\Bigg\} \end{aligned} \tag{45}$$

and

$$\langle a_{Q'}^* b^*\rangle_{H_{2\Lambda}}(\mu) = \frac{g_2}{2\sqrt{(\epsilon_{Q'} - \mu + \Omega)^2 - g_2^2}}\{(e^{\beta E_{2+}} - 1)^{-1} + (e^{\beta E_{2-}} - 1)^{-1} + 1\}. \tag{46}$$

If $\epsilon_{Q'} \geqslant g_2^2/4\Omega$ then $\mu_c(g_2, \Omega) = 0$ and we return to the PBG with no condensation in the Q'-mode. If $\epsilon_{Q'} < g_2^2/4\Omega$ we have to study two cases $g_2^2 \geqslant 4\Omega^2$ and $g_2^2 < 4\Omega^2$. Consider first the case $g_2^2 \geqslant 4\Omega^2$.

(a) *Case*: $\rho \leqslant \rho_c$ Proceeding in a similar manner as for the first model we obtain

$$\lim_{V\to\infty}\left\langle\frac{a^*_{Q'}a_{Q'}}{V}\right\rangle_{H_{2\Lambda}}(\mu_V) = \lim_{V\to\infty}\left\langle\frac{b^*b}{V}\right\rangle_{H_{2\Lambda}}(\mu_V) = \lim_{V\to\infty}\left\langle\frac{a^*_{Q'}b^*}{V}\right\rangle_{H_{2\Lambda}}(\mu_V) = 0. \tag{47}$$

(b) *Case*: $\rho > \rho_c$ As for the first case here we obtain

$$\lim_{V\to\infty}\left\langle\frac{a^*_{Q'}a_{Q'}}{V}\right\rangle_{H_{2\Lambda}}(\mu_V) = \rho - \rho_c \tag{48}$$

$$\lim_{V\to\infty}\left\langle\frac{b^*b}{V}\right\rangle_{H_{2\Lambda}}(\mu_V) = \frac{g_2^2}{4\Omega^2}(\rho - \rho_c) \tag{49}$$

and

$$\lim_{V\to\infty}\left\langle\frac{a^*_{Q'}b^*}{V}\right\rangle_{H_{2\Lambda}}(\mu_V) = \frac{g_2}{2\Omega}(\rho - \rho_c). \tag{50}$$

In the case $g_2^2 < 4\Omega^2$ we find that the limits have the same form but with g_2 and Ω interchanged.

3. Clearly in our models the superradiance (see e.g. [13, 14]) is directly related to BEC as is explicitly seen, for example, by comparing the formulae (33) and (34). Hence in spite of the simplicity of the models (5) they manifest an interesting cooperative phenomenon. The presence of the interaction between the Bose gas and radiation, compared to the PBG, occurs in both models at a lower critical density $\rho_c = \rho_0(\epsilon_Q - g^2/4\Omega) < \rho_{0c} = \rho_0(0)$. Moreover the condensation in these models takes place not only in dimension $d \geqslant 3$, but also in dimensions $d = 1$ and $d = 2$. This clearly shows that the presence of radiation enhances the process of condensation in the Bose gas.

It is also interesting to remark the value of the entangling boson–photon interaction energy in the two models, which one reads off from (35) and (50)

$$\lim_{V\to\infty}\left\langle\frac{U_{1,2\Lambda}}{V}\right\rangle_{H_{1,2\Lambda}}(\mu_V) = \mp\frac{g_{1,2}^2}{2\Omega}(\rho - \rho_c). \tag{51}$$

For the first model based on the *minimal coupling* we obtain the *negative* interaction energy (bound state) in the presence of condensates, which is well known [7–9, 14]. Whereas for the second model this interaction energy is *positive*, which is a completely different type of entanglement.

These aspects of our results make contact with the recent interest in entangled atom–photon states generated in BEC-superradiance experiments, see e.g. [1], addressed to a variety of applications such as tests of Bell inequalities, quantum cryptography and quantum teleportation.

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